

N^2 entropy of 4D $N = 4$ SYM

I. Y. Park

Center for Quantum Spacetime, Sogang University

Shinsu-dong 1, Mapo-gu, 121-742 South Korea

Department of Natural and Physical Sciences, Philander Smith College

Little Rock, AR 72223, USA

inyongpark05@gmail.com

Abstract

We employ localization technique to derive N^2 entropy scaling of four dimensional $N = 4$ SYM theory.

1 Introduction

Localization technique [1][2] has been proven to be powerful in current studies of supersymmetric gauge theories. It has been used recently in [3] and [4] to derive the $N^{3/2}$ and N^3 scalings of M2 and M5 branes respectively. It can bring a convenient redistribution of degrees of freedom over the various parts of the path integral when it comes to evaluating a partition function. This redistribution occurs via the choice of the localizing lagrangian and susy parameters. In this work, we consider four dimensional $N = 4$ SYM and employ localization technique to compute the entropy.¹

It has long been known that the near-extremal D3 brane supergravity solution has a Bekenstein-Hawking entropy that scales as N^2 [5]. Because of this, there was wide anticipation that it should be possible to observe the same scaling behavior from the SYM side. Although the N^2 behavior has been demonstrated using a free gas model [6], the same has not been achieved in the context of 4D SYM with interactions taken into account. (See [7] and [8] for related discussions.)

In the $N = 1$ description of $N = 4$ SYM (see, e.g., [9][10] for a review of supersymmetric gauge theories), there are three $N = 1$ chiral multiplets and one $N = 1$ vector multiplet. One of the three chiral multiplets belongs to the $N=2$ gauge multiplet and the other two belong to the $N=2$ hypermultiplet. All three chiral multiplets are in the adjoint representation, and on an equal footing in terms of the $N = 1$ description.

Even though localization technique² greatly simplifies the evaluation of partition functions in general, an explicit evaluation typically requires non-trivial computations. This is especially true when the computation involves instanton contributions; after introducing localization terms in the action and finding an extremum configuration, one expands the localizing action around the extremum configuration. The resulting expression is quadratic in the fluctuation fields but the computation is still not simple since the coefficients now involve the instanton configuration; the evaluation would

¹Our method should be applicable to other cases.

²One subtle issue in the enterprise of computing partitions functions is associated with Higgs vevs vs instanton moduli. When one considers the path integral one must not integrate over the Higgs vevs since they are physical "observables" whereas one integrates over the instanton moduli.

require the use of the index theorem. We show below that there is a localization procedure that requires minimal use of this theorem. (Moreover, the necessary use of index theorem is one that has already been known, as we will point out.) For this, we start with the observation that not all the formulations of $N = 4$ SYM are equally effective in evaluating the partition function: a formulation in which the susy transformations of the $N = 1$ chiral multiplets do not involve the fields in the $N = 1$ vector multiplet is much more effective.

Our strategy is as follows. We employ the $N = 4$ SYM formulation in which each $N = 1$ chiral multiplet transforms within itself, as was discussed in several textbooks. (The $N = 1$ vector multiplet also transforms within itself.) To be specific, we follow the notations and conventions of [10]. We take the chiral multiplet in the $N = 2$ gauge multiplet for an illustration of the localization of the chiral multiplets. As a result of the localization of the three chiral multiplets, the $N = 1$ gauge part of the lagrangian can be evaluated independently of the chiral multiplets. Combined with the result in literature, the localization leads to full evaluation of the partition function.

We speculate on the degrees of freedom that are responsible for the N^2 growth of the entropy at the end.

In one common of formulation of $N = 4$ SYM, the supersymmetry transformation of the chiral multiplet fermions involve gauge field, and this feature makes the evaluation of partition complicated. This is because the fluctuation fields couple to the instanton background, and one must perform the instanton sum at the final stage. Pleasantly enough, this complication can be avoided by using the $N=4$ SYM formulation that was discussed, e.g., in [10]. The key point is that the susy transformation of the $N=1$ chiral multiplets act within themselves in that formulation, and in particular they do not involve the gauge fields in the $N=1$ vector multiplet.

Below we will show that the localization of the three chiral supermultiplets leads to decoupling between the vector multiplet and the chiral multiplets. This implies that the vector multiplet part of the partition function can be evaluated separately. One can

again rely on localization technique for this evaluation. The evaluation consists of two parts: the "classical" part and the quadratic part around the instanton configurations. The classical part would give just the instanton numbers. The quadratic part was already evaluated through index theorem. (See, e.g., [11] and [12] and [13] for reviews.) The setup of [13] can be viewed as a part of the aforementioned localization procedure as we will discuss shortly.

The contribution of the $N = 1$ vector multiplet gives a trivial contribution as one can see as follows.³ It was reviewed in [13] that the one-loop partition function of the system is trivial. In fact, the result of [13] can be regarded as the full result by the following observation: The computation of [13] essentially contains localization procedure with a localizing term chosen such that it reproduces the entire $N = 1$ gauge action upon being acted by supersymmetry transformation. The existence of such a localizing term is guaranteed by the fact that the lagrangian would form a supermultiplet. Therefore, evaluation of the partition function of the $N = 1$ gauge action consists of the "classical part" and the quadratic part that results from localization. The classical part yields instanton numbers while the quadratic part yields a trivial result as shown in [13].

The three $N = 1$ chiral multiplets are on an equal footing in terms of the $N = 1$ description, and we illustrate the procedure with one of them. We consider the following form of the localization action⁴

$$Q\left(\overline{\psi_{AL}} \left[\sqrt{2} \partial_\nu \phi_A \gamma^\nu \alpha_L\right]\right) = \overline{(Q\psi_A)} \sqrt{2} \partial_\nu \phi_A \gamma^\nu \alpha_L + \overline{\psi_{AL}} \sqrt{2} \partial_\nu (Q\phi_A) \gamma^\nu \alpha_L \quad (1.2)$$

where

$$\alpha_L = \frac{1 + \gamma_5}{2} \alpha, \quad \alpha_R = \frac{1 - \gamma_5}{2} \alpha \quad (1.3)$$

and similarly for ψ_L, ψ_R ; the subscript A denotes the adjoint index. Q is a susy generator that includes a susy parameter spinor α , and the supersymmetry transformations

³This is particularly easy to see in the symmetric phase where all the scalar vevs are set to zero. It should also be true in the broken symmetry phases.

⁴Some of the discussions below become clearer by adding an addition localization term

$$Q\left(\overline{\psi_{AR}} \left[\sqrt{2} \partial_\nu \phi_A^* \gamma^\nu \alpha_R\right]\right) \quad (1.1)$$

of the chiral multiplet are⁵

$$\begin{aligned}\delta\psi_{AL} &= \sqrt{2} \partial_\mu \phi_A \gamma^\mu \alpha_R \\ \delta\phi_A &= \sqrt{2} \overline{\alpha_R} \psi_{AL}\end{aligned}\tag{1.4}$$

This choice brings certain redistribution of the degrees of freedom mentioned in the introduction, and is a crucial step in our localization procedure. One can show, after some algebra,

$$\overline{(Q\psi_A)} \sqrt{2} \partial_\nu \phi_A \gamma^\nu \alpha_L = 2 \partial_\mu \phi_A \partial^\mu \phi_A^* \left(\bar{\epsilon} \frac{1 + \gamma^5}{2} \epsilon \right)\tag{1.5}$$

We choose zero vevs for all of the scalars. The second term in (1.2) leads to

$$\overline{\psi_{AL}} \sqrt{2} \partial_\nu (Q\phi_A) \gamma^\nu \alpha_L = -\frac{1}{2} \overline{\psi_A} (1 - \gamma^5) \not{D}\psi_A \left(\bar{\epsilon} \frac{1 + \gamma^5}{2} \epsilon \right)\tag{1.6}$$

With this choice, therefore, the localization action takes

$$Q \left(\overline{\psi_{AL}} \sqrt{2} \partial_\nu \phi_A \gamma^\nu \alpha_L \right) = 2 \partial_\mu \phi_A \partial^\mu \phi_A^* \left(\bar{\epsilon} \frac{1 + \gamma^5}{2} \epsilon \right) - \frac{1}{2} \overline{\psi_A} (1 - \gamma^5) \not{D}\psi_A \left(\bar{\epsilon} \frac{1 + \gamma^5}{2} \epsilon \right)\tag{1.7}$$

Several remarks are in order. Compared with the standard form of the fermionic kinetic term, the projection operator $\frac{1 - \gamma^5}{2}$ is present in the action above. This indicates that half of the fermionic degrees of freedom are removed by the projection operator. Therefore a mismatch between the bosonic and fermionic degrees of freedom has arisen, and this disparity should be responsible for the non-vanishing entropy. The extremizing configurations are such that $\langle \phi \rangle = \text{const}$, $\psi = 0$. Let us consider the symmetric phase without scalar vevs: $\langle \phi \rangle = 0$. In that sector, the $N = 1$ gauge multiplet part of the action decouples from the chiral multiplet parts of the action. The N^2 -scaling trivially follows from the fact that these fields are in the adjoint representation: when the two adjoint indices get contracted they yield $N^2 - 1$ factor.

To compare with the case featuring the standard fermionic term, we first compute the partition function of the standard free system,

$$\mathcal{L} = \partial^\mu \phi_A \partial_\mu \phi_A^* + \frac{1}{2} (\bar{\psi}_A \not{D}\psi_A)\tag{1.8}$$

⁵Note that the term in the square bracket of the first term in (1.2) is the same as $\delta\psi_{AL}$ except for $\alpha_R \rightarrow \alpha_L$.

The partition needs to be regulated, and we choose a regularization that makes the analysis parallel to that of the finite temperature field theory⁶ in which the time direction is made periodic with periodicity L . The path integral yields

$$e^{-\frac{(N^2-1)3}{2} \sum_n \int \frac{d^3\vec{p}}{(2\pi)^3} \ln \left[(2\pi n)^2 L^2 + \vec{p}^2 \right]} e^{\frac{(N^2-1)3}{2} \sum_n \int \frac{d^3\vec{p}}{(2\pi)^3} \ln \left(\gamma^0 [i\gamma^0(2\pi n) + i\gamma^i p_i] \right)} \quad (1.9)$$

Let us evaluate the exponents:

$$h_B \equiv 3 \sum_n \int \frac{d^3\vec{p}}{(2\pi)^3} \ln \left[(2\pi n)^2 L^2 + \vec{p}^2 \right] \quad (1.10)$$

The factor 3 comes from the fact that there are three chiral multiplets. Consider

$$\frac{\partial}{\partial L} h_B = 24\pi^2 L \sum_n n^2 \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{(2\pi n)^2 L^2 + \vec{p}^2} \quad (1.11)$$

Let us turn to the fermionic part; define

$$h_F \equiv 3 \cdot \frac{1}{2} \sum_n \int \frac{d^3\vec{p}}{(2\pi)^3} \ln \left(\gamma^0 [i\gamma^0(2\pi nL) + i\gamma^i k_i] \right) = \frac{3}{2} \sum_n \int \frac{d^3\vec{p}}{(2\pi)^3} \ln \left(-i(2\pi nL) + i\gamma^0 \gamma^i k_i \right) \quad (1.12)$$

The factor 1/2 is due to the fact that the fermion is Majorana. Taking the L -derivative yields

$$\begin{aligned} \frac{\partial h_F}{\partial L} &\equiv \frac{3}{2} \text{Tr} \sum_n (-2\pi i n) \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\left(\gamma^0 [i\gamma^0(2\pi nL) + i\gamma^i k_i] \right)} \\ &= -24\pi^2 L \sum_n n^2 \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{(2\pi nL)^2 + \vec{k}^2} \end{aligned} \quad (1.13)$$

This cancels the bosonic contribution exactly as it should due to the supersymmetry. One can explicitly evaluate the values of the expressions that have appeared above. For example,

$$\frac{\partial}{\partial L} h_B = -12\pi^2 L^2 \zeta(-3) = -\frac{\pi^2}{10} L^2 \quad (1.14)$$

⁶Other regularizations should be possible; for example, one may put the system on a sphere. One would still find the N^2 behavior although the overall numerical coefficient would be different.

The equality follows from the combined use of dimensional regularization and zeta function regularization. ζ denotes the Riemann zeta function, and we have used $\zeta(-3) = \frac{1}{120}$. To get the L -dependent part of the entropy, one should integrate (1.13) and take one half of the value (which is due to the presence of the projection operator in (1.7)) after removing the minus sign. The net entropy turns out to be⁷

$$\frac{\pi^2}{60} L^3 N^2 \tag{1.15}$$

Finally we comment on the degrees of freedom that should be responsible for the entropy. It is likely to be the goldstino multiplet of the SYM counterparts of the supergravity $\frac{1}{16}$ -BPS states that are behind the N^2 behavior. Some discussions on $\frac{1}{16}$ -BPS states can be found in [14] and [15].

Acknowledgements

I am grateful to M. Rocek for the valuable discussions. I thank E. Hatefi for his hospitality during my visit to ICTP, Trieste.

⁷As commented in footnote 4, one may add additional localization term. Then susy parameter factors would take the form of $(\bar{\epsilon}\frac{1-\gamma^5}{2}\epsilon)$, and it is not possible to re-scale both types of the factors away since the fermions are of Majorana type.

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